Assignment 6

April 11, 2017

Exercise 6.1: 2, 4, 6, 7, 9, 11
Exercise 6.2: 1, 2, 3, 4, 6, 7(a)
Exercise 6.3: 1, 2, 3

Problem 4. Let $u \geq 0$ and $\Delta u = 0$ in a unit disk $D = \{(x,y)|x^2 + y^2 \leq 1\}$. Using the Mean-Value Property to prove the following so-called Harnack inequality

$$\frac{1-r}{1+r}u(0,0) \leq u(x,y) \leq \frac{1+r}{1-r}u(0,0)$$

where $r = \sqrt{x^2 + y^2} < 1$.

Problem 5. Consider the following problem

$$\begin{cases} \Delta u = 0 & \text{in } D = \{x^2 + y^2 \leq 1\} \\ u = h & \text{on } \partial D \end{cases} \tag{1}$$

(a) Show that if $h \geq 0$, then $u > 0$ in D unless $h = 0$.

(b) Let $u(0) = 1$ and $h \geq 0$. Show that

$$\frac{1}{3} \leq u(x,y) \leq 3$$

for all $x^2 + y^2 = \frac{1}{4}$

Problem 6. Suppose that u satisfies $u_{xx} + u_{yy} = 0$ for all $(x,y) \in B_1(0)$ except $(x,y) = (0,0)$. Show that if u is bounded, then $\lim_{(x,y)\to(0,0)} u(x,y)$ exists and by taking $u(0,0) = \lim_{(x,y)\to(0,0)} u(x,y)$, u is actually smooth in $B_1(0)$.

Hint: Consider the following function $v_\epsilon = \epsilon \log \frac{1}{r}$.

Exercise 6.4: 1, 6, 10, 11, 13

Probme 7. Using the method of separation of variables to solve the following problem

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & \text{in } D = \{(r,\theta)|1 < r < 2, 0 \leq \theta \leq \pi\} \\ u(1,\theta) = \cos^3(\frac{\theta}{2}), u(2,\theta) = 4\cos(\frac{5\theta}{2}) & \\ u_\theta(r,0) = 0, u(r,\pi) = 0. & \end{cases} \tag{2}$$

Exercise 6.1

2. Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$, where k is a positive constant. (*Hint:* Substitute $u = v/r$.)

4. Solve $u_{xx} + u_{yy} + u_{zz} = 0$ in the spherical shell $0 < a < r < b$ with the boundary conditions $u = A$ on $r = a$ and $u = B$ on $r = b$, where A and B are constants. (*Hint:* Look for a solution depending only on r.)

6. Solve $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b$ with $u(x, y)$ vanishing on both parts of the boundary $r = a$ and $r = b$.

7. Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell $a < r < b$ with $u(x, y, z)$ vanishing on both the inner and outer boundaries.

9. A spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. Its inner boundary is held at 100 °C. Its outer boundary satisfies $\partial u/\partial r = -\gamma < 0$, where γ is a constant.
   (a) Find the temperature. (*Hint:* The temperature depends only on the radius.)
   (b) What are the hottest and coldest temperatures?
   (c) Can you choose γ so that the temperature on its outer boundary is 20 °C?

11. Show that there is no solution of
$$\Delta u = f \quad \text{in } D, \qquad \frac{\partial u}{\partial n} = g \quad \text{on bdy } D$$
in three dimensions, unless
$$\iiint_D f\,dxdydz = \iint_{\text{bdy}(D)} g\,dS.$$
(*Hint:* Integrate the equation.) Also show the analogue in one and two dimensions.

Exercise 6.2

1. Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < a$, $0 < y < b$ with the following boundary conditions:
$$u_x = -a \quad \text{on } x = 0 \qquad\qquad u_x = 0 \quad \text{on } x = a$$
$$u_y = b \quad \text{on } y = 0 \qquad\qquad u_y = 0 \quad \text{on } x = b.$$
(*Hint:* Note that the necessary condition of Exercise 6.1.11 is satisfied. A shortcut is to guess that the solution might be a quadratic polynomial in x and y.)

2. Prove that the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$.

3. Find the harmonic function $u(x, y)$ in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions:
$$u_y = 0 \qquad \text{for } y = 0 \text{ and for } y = \pi,$$
$$u = 0 \qquad \text{for } x = 0,$$
$$u = \cos y^2 = \frac{1}{2}(1 + \cos 2y) \qquad \text{for } x = \pi.$$

4. Find the harmonic function in the square $\{0 < x < 1, 0 < y < 1\}$ with the boundary conditions $u(x, 0) = x$, $u(x, 1) = 0$, $u_x(0, y) = 0$, $u_x(1, y) = y^2$.

6. Solve the following Neumann problem in the cube $\{0 < x < 1, 0 < y < 1, 0 < z < 1\}$: $\Delta u = 0$ with $u_z(x, y, 1) = g(x, y)$ and homogeneous Neumann conditions on the other five faces, where $g(x, y)$ is an arbitrary function with zero average.

7(a). Find the harmonic function in the semi-infinite strip $\{0 \le x \le \pi, 0 \le y < \infty\}$ that satisfies the "boundary conditions":

$$u(0, y) = u(\pi, y) = 0,\ u(x, 0) = h(x),\ \lim_{y\to\infty} u(x, y) = 0.$$

Exercise 6.3

1. Suppose that u is a harmonic function in the disk $D = \{r < 2\}$ and that $u = 3\sin 2\theta + 1$ for $r = 2$. Without finding the solution, answer the following questions.
   (a) Find the maximum value of u in $\bar{D}$.
   (b) Calculate the value of u at the origin.

2. Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition

$$u = 1 + 3\sin\theta \quad \text{on } r = a.$$

3. Same for the boundary condition $u = \sin^3\theta$. (*Hint:* Use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.)

Exercise 6.4

1. Solve $u_{xx} + u_{yy} = 0$ in the *exterior* $\{r > a\}$ of a disk, with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$, and the condition at infinity that u be bounded as $r \to \infty$.

6. Find the harmonic function u in the semidisk $\{r < 1,\ 0 < \theta < \pi\}$ with u vanishing on the diameter $(\theta = 0, \pi)$ and

$$u = \pi\sin\theta - \sin 2\theta \quad \text{on } r = 1.$$

10. Solve $u_{xx} + u_{yy} = 0$ in the quarter-disk $\{x^2 + y^2 < a^2, x > 0, y > 0\}$ with the following BCs:

$$u = 0 \quad \text{on } x = 0 \text{ and on } y = 0, \qquad \text{and } \frac{\partial u}{\partial r} = 1 \quad \text{on } r = a.$$

Write the answer as an infinite series and write the first two nonzero terms explicitly.

11. Prove the uniqueness of the Robin problem

$$\Delta u = f \text{ in } D,\ \frac{\partial u}{\partial n} + au = h \quad \text{on bdy } D,$$

where D is any domain in three dimensions and where a is a positive constant.

13. Solve $u_{xx} + u_{yy} = 0$ in the region $\{\alpha < \theta < \beta,\ a < r < b\}$ with the boundary conditions $u = 0$ on the two sides $\theta = \alpha$ and $\theta = \beta$, $u = g(\theta)$ on the arc $r = a$, and $u = h(\theta)$ on the arc $r = b$.